A UNIFORMLY CONTINUOUS LINEAR EXTENSION PRINCIPLE IN TOPOLOGICAL VECTOR SPACES WITH AN APPLICATION TO LEBESGUE INTEGRATION

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Abstract. We prove a uniformly continuous linear extension principle in topological vector spaces from which we derive a very short and canonical construction of the Lebesgue integral of Banach space valued maps on a finite measure space. The Vitali Convergence Theorem and the Riesz-Fischer Theorem follow as easy consequences from our construction.

1. Introduction and motivation

Since the birth of Lebesgue's theory of integration, various introductions and modifications of the theory have been proposed in an attempt to

(1) make it more elementary ([9]),
(2) unify it with Riemann's approach to integration ([5],[6],[8],[11]),
(3) extend it to a Banach space valued context ([1],[4],[10]),
(4) present it in the abstract setting of functional analysis ([3],[7]).

In this paper we shall present a very short and canonical construction of the Lebesgue integral via a uniformly continuous linear extension principle in topological vector spaces (Theorem 2.2) which addresses most of the above mentioned topics. The Vitali Convergence Theorem and the Riesz-Fischer Theorem are shown to follow as easy consequences from our construction.

2. Uniformly continuous linear extension in TVS

The following lemma is a well known result in the theory of uniform spaces, see e.g. [2].

Lemma 2.1. *Let X be a uniform space, $A \subset X$ a dense subset and f a uniformly continuous map of A into a complete Hausdorff uniform space. Then there exists a unique uniformly continuous extension of f to X.*

We say that a collection $\mathcal{K}$ of subsets of a complex vector space is *closed under the formation of finite linear combinations* iff the set $\alpha K_1 + \beta K_2 = \{\alpha x + \beta y \mid x \in K_1, y \in K_2\}$ belongs to $\mathcal{K}$ for all $K_1, K_2 \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$. We now apply Lemma 2.1 to obtain the following uniformly continuous linear extension principle in complex topological vector spaces (TVS).

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Theorem 2.2. *Let E be a TVS, $F \subset E$ a vector subspace, $\mathcal{K}$ a collection of sets $K \subset F$ which covers F and is closed under the formation of finite linear combinations. Let λ be a linear map of F into a complete Hausdorff TVS E' which is uniformly continuous on each $K \in \mathcal{K}$. Then $\widetilde{F} = \cup_{K \in \mathcal{K}} \overline{K}$ is a vector space containing F and there exists a unique linear extension of λ to $\widetilde{F}$ which is uniformly continuous on the closure of each $K \in \mathcal{K}$.*

Proof. For $x, y \in \widetilde{F}$ and $\alpha, \beta \in \mathbb{C}$, choose $K_1, K_2 \in \mathcal{K}$ such that $x \in \overline{K_1}$ and $y \in \overline{K_2}$. Then $\alpha x + \beta y \in \alpha\overline{K_1} + \beta\overline{K_2} \subset \overline{\alpha K_1 + \beta K_2} \subset \widetilde{F}$, the latter inclusion being a consequence of the fact that $\mathcal{K}$ is closed under the formation of finite linear combinations. We conclude that $\widetilde{F}$ is a vector space, which contains F because $\mathcal{K}$ covers F. Furthermore, for each $K \in \mathcal{K}$, Lemma 2.1 shows that $\lambda\mid_K$ extends uniquely to a uniformly continuous map $\overline{\lambda\mid_K}$ of $\overline{K}$ into E'. For $x \in \widetilde{F}$, choose $K \in \mathcal{K}$ such that $x \in \overline{K}$ and put $\widetilde{\lambda}(x) = \overline{\lambda\mid_K}(x)$. Then $\widetilde{\lambda}$ is the desired extension. □

3. Construction of the Lebesgue integral

Let $\Omega = (\Omega, \mathcal{A}, \mu)$ be a finite measure space, $E = (E, \|\cdot\|)$ a complex Banach space and $\mathbf{M}(\Omega, E)$ the TVS of Borel measurable maps f of Ω into E, equipped with the topology of convergence in measure. That is, the sets

$$\mathbf{V}_\epsilon = \{f \in \mathbf{M}(\Omega, E) \mid \mu(\{\|f\| \geq \epsilon\}) < \epsilon\}, \ \ \epsilon > 0,$$

constitute a base for the neighbourhood filter of 0. It is well known that the uniform structure of $\mathbf{M}(\Omega, E)$ is complete and pseudometrisable, see e.g. [3]. Unless otherwise stated, all subsets of $\mathbf{M}(\Omega, E)$ are equipped with the uniformity of convergence in measure.

A map $s \in \mathbf{M}(\Omega, E)$ is called *simple* iff there exists a finite measurable partition $A_1, \ldots, A_n$ of Ω such that s assumes a unique value $s_i \in E$ on each A_i. We denote the collection of simple maps as $\mathbf{S}(\Omega, E)$. Notice that $\mathbf{S}(\Omega, E)$ is a vector subspace of $\mathbf{M}(\Omega, E)$. We define the *integral of* $s \in \mathbf{S}(\Omega, E)$ as $\int s = \sum_i \mu(A_i)s_i$, with $A_1, \ldots, A_n$ the measurable partition and $s_1, \ldots, s_n$ the values associated with s.

Proposition 3.1. *The mapping $\int$ of $\mathbf{S}(\Omega, E)$ into E is linear and, if $E = \mathbb{C}$, positive in the sense that $\int s \geq 0$ if $s \geq 0$. Also, if $s \in \mathbf{S}(\Omega, E)$, then $\|s\| \in \mathbf{S}(\Omega, \mathbb{C})$ and $\|\int s\| \leq \int \|s\|$.*

Proof. This is standard. □

We call a set $\mathbf{E} \subset \mathbf{S}(\Omega, E)$ *elementary* iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $\int \|s\| 1_A < \epsilon$ whenever $s \in \mathbf{E}$ and $A \in \mathcal{A}$ with $\mu(A) < \delta$. The collection of elementary sets in $\mathbf{S}(\Omega, E)$ is denoted as $\mathcal{E}(\Omega, E)$.

Proposition 3.2. *$\mathcal{E}(\Omega, E)$ contains all sets $\mathbf{E} \subset \mathbf{S}(\Omega, E)$ which, equipped with the weak uniformity for the mapping $\int \circ \|\cdot\|$ of $\mathbf{E}$ into $\mathbb{C}$, are totally bounded, and is closed under the formation of finite linear combinations.*

Proof. If $\mathbf{E} \subset \mathbf{S}(\Omega, E)$ is finite, then there exists a constant $C > 0$ such that $\|s\| \leq C$ for all $s \in \mathbf{E}$, whence $\mathbf{E} \in \mathcal{E}(\Omega, E)$, and the first assertion easily follows. The second assertion follows from Proposition 3.1. □

Proposition 3.3. *The uniformity of convergence in measure is weaker than the weak uniformity for the mapping $\int \circ \|\cdot\|$ of $\mathbf{S}(\Omega, E)$ into $\mathbb{C}$, and these uniformities coincide on elementary sets. In particular, the mapping $\int$ of $\mathbf{S}(\Omega, E)$ into E is uniformly continuous on elementary sets.*

Proof. Observe that

$$\mu(\{\|s-t\| \geq \epsilon\}) \leq \epsilon^{-1} \int \|s-t\|, \tag{1}$$

$$\int \|s-t\| \leq \epsilon\mu(\Omega) + \int \|s-t\| 1_{\{\|s-t\|\geq\epsilon\}}, \tag{2}$$

$$\left\| \int s - \int t \right\| \leq \int \|s-t\| \tag{3}$$

for all $s, t \in \mathbf{S}(\Omega, E)$ and $\epsilon > 0$. □

A map $f \in \mathbf{M}(\Omega, E)$ is called *(Lebesgue) integrable* iff it belongs to $\mathbf{L}(\Omega, E) = \cup_{\mathbf{E}\in\mathcal{E}(\Omega,E)} \overline{\mathbf{E}}$. From Theorem 2.2 we conclude that $\mathbf{L}(\Omega, E)$ is a vector space containing $\mathbf{S}(\Omega, E)$ and that there exists a unique linear extension of $\int$ to $\mathbf{L}(\Omega, E)$ which is uniformly continuous on the closure of each elementary set. We denote this extension again as $\int$ and we define the *integral of* $f \in \mathbf{L}(\Omega, E)$ as $\int f$.

Proposition 3.4. *Proposition 3.1 continues to hold if we replace* $\mathbf{S}(\Omega, E)$ *by* $\mathbf{L}(\Omega, E)$ *and* $\mathbf{S}(\Omega, \mathbb{C})$ *by* $\mathbf{L}(\Omega, \mathbb{C})$.

Proof. This follows easily from the fact that $\int$ is continuous on the closure of each elementary set. □

We call a set $\mathbf{F} \subset \mathbf{L}(\Omega, E)$ *uniformly integrable* iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $\int \|f\| 1_A < \epsilon$ whenever $f \in \mathbf{F}$ and $A \in \mathcal{A}$ with $\mu(A) < \delta$. Notice that the closure of an elementary set is uniformly integrable because $\int$ is continuous on such a set. The collection of uniformly integrable sets in $\mathbf{L}(\Omega, E)$ is denoted as $\mathcal{I}(\Omega, E)$.

Proposition 3.5. *Proposition 3.2 continues to hold if we replace* $\mathcal{E}(\Omega, E)$ *by* $\mathcal{I}(\Omega, E)$ *and* $\mathbf{S}(\Omega, E)$ *by* $\mathbf{L}(\Omega, E)$.

Proof. A finite subset of $\mathbf{L}(\Omega, E)$ is uniformly integrable because it is contained in the closure of an elementary set. The rest of the proof is analogous to the proof of Proposition 3.2. □

Proposition 3.6. *Proposition 3.3 continues to holds if we replace* $\mathbf{S}(\Omega, E)$ *by* $\mathbf{L}(\Omega, E)$ *and 'elementary set' by 'uniformly integrable set'.*

Proof. This is analogous to the proof of Proposition 3.3. □

Corollary 3.7. *The space* $\mathbf{L}(\Omega, E)$*, equipped with the weak uniformity for the mapping* $\int \circ \|\cdot\|$ *of* $\mathbf{L}(\Omega, E)$ *into* $\mathbb{C}$*, contains* $\mathbf{S}(\Omega, E)$ *as a dense subspace.*

Proof. This follows immediately from Proposition 3.6. □

Corollary 3.8. *(Vitali) Fix* $f \in \mathbf{M}(\Omega, E)$ *and* $(f_n)_n$ *in* $\mathbf{L}(\Omega, E)$*. Then the following are equivalent.*

(1) $f \in \mathbf{L}(\Omega, E)$ *and* $\int \|f - f_n\| \to 0$.

(2) $\{f_n \mid n\} \in \mathcal{I}(\Omega, E)$ *and* $f_n \xrightarrow{\mu} f$.

Proof. (1) $\Rightarrow$ (2) This follows immediately from Propositions 3.5 and 3.6. (2) $\Rightarrow$ (1) Corollary 3.7 allows us to choose, for $n \in \mathbb{N}_0$, $s_n \in \mathbf{S}(\Omega, E)$ such that $\int \|f_n - s_n\| \leq 1/n$. Now $\{s_n \mid n\} \in \mathcal{E}(\Omega, E)$, whence $f \in \overline{\{s_n \mid n\}} \subset \mathbf{L}(\Omega, E)$, and Proposition 3.6 reveals that $\int \|f - f_n\| \to 0$. $\square$

Corollary 3.9. *(Riesz-Fischer) The space* $\mathbf{L}(\Omega, E)$*, equipped with the weak uniformity for the mapping* $\int \circ \|\cdot\|$ *of* $\mathbf{L}(\Omega, E)$ *into* $\mathbb{C}$*, is complete.*

Proof. Let $(f_n)_n$ be Cauchy in $\mathbf{L}(\Omega, E)$. It follows from Proposition 3.5 that $\{f_n \mid n\}$ is uniformly integrable and from Proposition 3.6 that $(f_n)_n$ is Cauchy, and thus convergent, in $\mathbf{M}(\Omega, E)$. An application of Corollary 3.8 completes the proof. $\square$

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